Generalizing Ramanujan's trigonometric identity

V.Davydov *

August 2017

Abstract

Let $p \equiv 1 \bmod 6$ be a prime number and let g be a primitive root mod p; put

$$S_p(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k}\right), \quad S_p'(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k+1}\right), \quad S_p''(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k+2}\right).$$

We present the following generalization of Ramanujan's trigonometric identity:

$$\sqrt[3]{S_p(g)} + \sqrt[3]{S_p'(g)} + \sqrt[3]{S_p''(g)} = \sqrt[3]{3\sqrt[3]{mp} - (6m+1)} \quad \text{whenever } p = 9m^2 + 3m + 1 \text{ for some } m \in \mathbb{Z}.$$

Using elliptic curves we prove that there are infinitely many pairs of rational numbers (α, β) such that

$$\sqrt[3]{\alpha - S_p(g)} + \sqrt[3]{\alpha - S_p'(g)} + \sqrt[3]{\alpha - S_p''(g)} = \sqrt[3]{\beta}.$$

Introduction and main results

In 1913 Ramanujan found the following trigonometric identity [1]

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{6\pi}{7}} = \sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}, \tag{1}$$

which appears in many different places; see, e.g., [BCK, p.7], [Mar, p.218], [Pr, p.104], [Sh, p.52].

In 1968 Levin [Lev, p.31-32] conjectured that (1) is just a particular case of much more general identity:

> "Эти точные равенства являются, конечно, частными случаями значительно более общих соотношений, которыми располагал Рамануджан, но о которых он никому ничего не сообщил. После его смерти часть этих общих соотношений была восстановлена другими математиками, но не подлежит сомнению, что некоторые из них утеряны навсегда".

We show that this is indeed the case—namely, identity (2) below generalizes (1).

In 2005 Markelov [Mar, p.219] obtained three other formulas with denominators 13,31,43 instead of 7. So one can ask why these numbers. Nothing easier, here's an explanation:

$$7 = 9\cdot(-1)^2 + 3\cdot(-1) + 1, \quad 13 = 9\cdot 1^2 + 3\cdot 1 + 1, \quad \text{(Each number is of the form}$$
$$31 = 9\cdot(-2)^2 + 3\cdot(-2) + 1, \quad 43 = 9\cdot 2^2 + 3\cdot 2 + 1. \quad 9m^2 + 3m + 1 \text{ with } m \in \mathbb{Z}.)$$

Note that Markelov used some kind of trigonometric sums, but a priori it is not clear how to give a definition in the general case. Nevertheless, there is a way to overcome this obstacle.

Let $p \equiv 1 \bmod 6$ be a prime number and let g be a primitive root mod p, that is, a generator of the cyclic group $\mathbb{F}_p^\times \cong C_{p-1}$; put

$$S_p(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k}\right), \quad S_p'(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k+1}\right), \quad S_p''(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k+2}\right).$$

The problem is that we don't know much about these sums, and presumably you don't find any reference in the current literature. Fortunately for us, it turns out that $S_p(g), S_p'(g), S_p''(g)$ are closely related with so-called *cubic Gauss sums*, see Lemma 1.1 below. Eventually, this connection will allow us to establish the following result.

Theorem 1. *With $S_p(g), S_p'(g), S_p''(g)$ as above, suppose that $p = 9m^2 + 3m + 1$ for some $m \in \mathbb{Z}$; then we have*

$$\sqrt[3]{S_p(g)} + \sqrt[3]{S_p'(g)} + \sqrt[3]{S_p''(g)} = \sqrt[3]{3\sqrt[3]{mp} - (6m+1)}. \tag{2}$$

*e-mail: davidoff303@yandex.ru

[1] Throughout this paper the sign $\sqrt[3]{\ }$ means a *real–valued* cube root.

Now is the time to give some examples.

- Let $m = -1$, then we get

$$\sqrt[3]{\sum_{k=0}^{1} \cos\left(\frac{2\pi}{7} \cdot 3^{3k}\right)} + \sqrt[3]{\sum_{k=0}^{1} \cos\left(\frac{2\pi}{7} \cdot 3^{3k+1}\right)} + \sqrt[3]{\sum_{k=0}^{1} \cos\left(\frac{2\pi}{7} \cdot 3^{3k+2}\right)} = \sqrt[3]{5 - 3\sqrt[3]{7}},$$

which is equivalent to identity (1).

- Let $m = 1, -2, 2$, then we get

$$\sqrt[3]{\sum_{k=0}^{3} \cos\left(\frac{2\pi}{13} \cdot 2^{3k}\right)} + \sqrt[3]{\sum_{k=0}^{3} \cos\left(\frac{2\pi}{13} \cdot 2^{3k+1}\right)} + \sqrt[3]{\sum_{k=0}^{3} \cos\left(\frac{2\pi}{13} \cdot 2^{3k+2}\right)} = \sqrt[3]{-7 + 3\sqrt[3]{13}},$$

$$\sqrt[3]{\sum_{k=0}^{9} \cos\left(\frac{2\pi}{31} \cdot 3^{3k}\right)} + \sqrt[3]{\sum_{k=0}^{9} \cos\left(\frac{2\pi}{31} \cdot 3^{3k+1}\right)} + \sqrt[3]{\sum_{k=0}^{9} \cos\left(\frac{2\pi}{31} \cdot 3^{3k+2}\right)} = \sqrt[3]{11 - 3\sqrt[3]{62}},$$

$$\sqrt[3]{\sum_{k=0}^{13} \cos\left(\frac{2\pi}{43} \cdot 3^{3k}\right)} + \sqrt[3]{\sum_{k=0}^{13} \cos\left(\frac{2\pi}{43} \cdot 3^{3k+1}\right)} + \sqrt[3]{\sum_{k=0}^{13} \cos\left(\frac{2\pi}{43} \cdot 3^{3k+2}\right)} = \sqrt[3]{-13 + 3\sqrt[3]{86}},$$

which are equivalent to Markelov's formulas in [Mar, p.219]; see Table 1 for more examples.

Meanwhile, the expression in the right-hand side of (1) can be further simplified,

$$\sqrt[3]{\frac{74}{43} + 2\cos\frac{2\pi}{7}} + \sqrt[3]{\frac{74}{43} + 2\cos\frac{4\pi}{7}} + \sqrt[3]{\frac{74}{43} + 2\cos\frac{6\pi}{7}} = \sqrt[3]{\frac{392}{43}}.$$

This identity, as well as plenty of others, was founded in 2016 by the present author (the key point was to realize that the equation $Y^3 = X^3 + X^2 - 2X - 1$ can be transformed into an elliptic curve of rank 1). After this, mercio constructed a rational 3-isogeny between two elliptic curves which are given by $(Y+2)(Y+9) = X^3$ and $Y^3 = X^3 + 7$, and then he showed that there are infinitely many pairs of rational numbers (α, β) such that

$$\sqrt[3]{\alpha + 2\cos\frac{2\pi}{7}} + \sqrt[3]{\alpha + 2\cos\frac{4\pi}{7}} + \sqrt[3]{\alpha + 2\cos\frac{6\pi}{7}} = \sqrt[3]{\beta};$$

see [ms1842663] for details. Theorem 2 generalizes this to arbitrary prime number $p \equiv 1 \bmod 6$.

Theorem 2. *Let $S_p(g), S'_p(g), S''_p(g)$ be as above; then there are infinitely many pairs of rational numbers (α, β) such that*

$$\sqrt[3]{\alpha - S_p(g)} + \sqrt[3]{\alpha - S'_p(g)} + \sqrt[3]{\alpha - S''_p(g)} = \sqrt[3]{\beta}.$$

Contents

1 Trigonometric sums

1.1 Proof of Theorem 1 modulo Lemma 1.1

Recall that any homomorphism from a group to the non-zero complex numbers is called *a character* of this group. Henceforth assume that p is a prime number such that $p \equiv 1 \bmod 6$; then the group $\mathbb{F}_p^\times$ has exactly two characters of order 3. For any such a character χ put

$$\vartheta_p(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=1}} \zeta^x, \quad \vartheta'_p(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=\omega}} \zeta^x, \quad \vartheta''_p(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=\omega^2}} \zeta^x.$$

Here ζ equals $e^{2\pi i/p}$ and ω equals $e^{2\pi i/3}$, see [Has, p.460]. By the way, the numbers

$$\eta_p(\chi) := 3\vartheta_p(\chi)+1, \quad \eta_p'(\chi) := 3\vartheta_p'(\chi)+1, \quad \eta_p''(\chi) := 3\vartheta_p''(\chi)+1$$

are usually called *cubic Gauss sums* (or sometimes *Kummer sums*). The main idea is to use the fact that $S_p(g)$, $S_p'(g)$, $S_p''(g)$ and $\vartheta_p(\chi)$, $\vartheta_p'(\chi)$, $\vartheta_p''(\chi)$ are "almost the same"; to make this precise, we present the following result.

Lemma 1.1. *For any character χ of order 3 and any primitive root g either*

$$S_p(g) = \vartheta_p(\chi), \quad S_p'(g) = \vartheta_p'(\chi), \quad S_p''(g) = \vartheta_p''(\chi)$$

or

$$S_p(g) = \vartheta_p(\chi), \quad S_p'(g) = \vartheta_p''(\chi), \quad S_p''(g) = \vartheta_p'(\chi).$$

Bearing this statement in mind, we now briefly recall some relevant facts from number theory.

By a well-known theorem of Gauss, for any prime number $p \equiv 1 \bmod 6$ there exist $a, b \in \mathbb{Z}$ such that $4p = a^2 + 27b^2$; moreover a, b are *uniquely determined* provided that $a \equiv 1 \bmod 3$ and $b > 0$, see [AR, Chapter 8] or any other standard reference. Furthermore, it is known that $\vartheta_p(g)$, $\vartheta_p'(g)$, $\vartheta_p''(g)$ are roots of the polynomial

$$h(\vartheta) := \vartheta^3 + \vartheta^2 - \frac{p-1}{3}\vartheta - \frac{ap+3p-1}{27} \in \mathbb{Z}[\vartheta],$$

see [Has, p.461]. In general, the real number $\left(\sqrt[3]{\vartheta_p(g)} + \sqrt[3]{\vartheta_p'(g)} + \sqrt[3]{\vartheta_p''(g)}\right)^3$ is a root of degree 9 polynomial over $\mathbb{Q}$ with solvable Galois group, i.e., it can be written as an expression in radicals; but it is rather complicated, see [dxdy54521] for the case when $p = 19$. However, we may put certain additional restriction on the coefficients of $h(\vartheta)$ in order to avoid unnecessary complications. The following lemma tell us what this additional restriction should look like.

Lemma 1.2. *Let t_1, t_2, t_3 be real numbers such that $\prod_{i=1}^{3}(t - t_i) = t^3 + a_1 t^2 + a_2 t + a_3 \in \mathbb{Q}[t]$; put*

$$b_1 := a_1 + 6\sqrt[3]{a_3}, \quad b_2 := a_1^2 + 3a_1\sqrt[3]{a_3} + 9(\sqrt[3]{a_3})^2 - 9a_2, \quad b_3 := (a_1 - 3\sqrt[3]{a_3})^3.$$

Suppose that $b_1^2 = b_2$ or equivalently, that

$$3 \cdot (\sqrt[3]{a_3})^2 + a_1\sqrt[3]{a_3} + a_2 = 0;$$

then

$$\sqrt[3]{t_1} + \sqrt[3]{t_2} + \sqrt[3]{t_3} = \sqrt[3]{\sqrt[3]{b_1^3 - b_3} - b_1}.$$

PROOF. See [Pr, Propositions 3 – 4]; note that our hypotheses are slightly different, but the proof can be done in a similar fashion.

□

We want to apply this result to the polynomial $h(\vartheta)$. In the above notation, we have

$$a_1 = 1, \quad a_2 = -\frac{p-1}{3}, \quad a_3 = -\frac{ap+3p-1}{27};$$

so it makes sense to ask when the equality

$$3 \cdot \left(\sqrt[3]{-\frac{ap+3p-1}{27}}\right)^2 + \sqrt[3]{-\frac{ap+3p-1}{27}} - \frac{p-1}{3} = 0$$

holds. Now we give a criterion.

Lemma 1.3. *Given a prime number $p \equiv 1 \bmod 6$ and the representation $4p = a^2 + 27b^2$ with $a \equiv 1 \bmod 3$ and $b > 0$. Then the following are equivalent:*

(*i*) $3 \cdot \left(\sqrt[3]{-\frac{ap+3p-1}{27}}\right)^2 + \sqrt[3]{-\frac{ap+3p-1}{27}} - \frac{p-1}{3} = 0.$

(*ii*) $p = 9m^2 + 3m + 1$ *for some* $m \in \mathbb{Z}$.

PROOF. Suppose that (i) holds; then an integer number $-\frac{ap+3p-1}{27}$ is necessarily a perfect cube, since otherwise 1 , $\sqrt[3]{-\frac{ap+3p-1}{27}}$, $\left(\sqrt[3]{-\frac{ap+3p-1}{27}}\right)^2$ are linearly independent over $\mathbb{Q}$ and therefore (i) is impossible. Thus, even $\sqrt[3]{-\frac{ap+3p-1}{27}}$ is an integer number, say m, and (i) can be written as $3m^2+m-\frac{p-1}{3}=0$; from this (ii) follows.

Conversely, suppose that $p=9m^2+3m+1$ for some $m\in\mathbb{Z}$; this implies that $4p=(-3m-2)^2+27|m|^2$. By the uniqueness of the representation $4p=a^2+27b^2$ (with $a\equiv 1 \bmod 3$ and $b>0$), we have

$$a=-3m-2 \quad \text{and} \quad b=|m|.$$

Substituting $9m^2+3m+1$ for p and $-3m-2$ for a, we find

$$-\frac{ap+3p-1}{27}=m^3.$$

We may now write (i) in the form $3m^2+m-\frac{p-1}{3}=0$, which indeed holds since $p=9m^2+3m+1$. $\square$

PROOF OF THEOREM 1. Since $p=9m^2+3m+1$ by hypothesis of the theorem, it follows that $\frac{p-1}{3}=m(3m+1)$; it also follows from the proof of Lemma 1.3 that $-\frac{ap+3p-1}{27}=m^3$. Therefore $h(\vartheta)$ takes the form

$$h(\vartheta)=\vartheta^3+\vartheta^2-m(3m+1)\vartheta+m^3.$$

In the notation of Lemma 1.2, we have

$$a_1=1, \quad a_2=-m(3m+1), \quad a_3=m^3;$$

hence

$$b_1=1+6m, \quad b_2=36m^2+12m+1, \quad b_3=(1-3m)^3.$$

As the equality

$$3\cdot(\sqrt[3]{a_3})^2+a_1\sqrt[3]{a_3}+a_2=0$$

holds, one can apply Lemma 1.2 to the polynomial $h(\vartheta)$; recalling that $\vartheta_p(g)$, $\vartheta'_p(g)$, $\vartheta''_p(g)$ are roots of $h(\vartheta)$, we obtain

$$\sqrt[3]{\vartheta_p(g)}+\sqrt[3]{\vartheta'_p(g)}+\sqrt[3]{\vartheta''_p(g)}=\sqrt[3]{\sqrt[3]{b_1^3-b_3}-b_1}.$$

Replacing b_1 by $1+6m$ and b_3 by $(1-3m)^3$, we get

$$\sqrt[3]{\sqrt[3]{b_1^3-b_3}-b_1}=\sqrt[3]{3\sqrt[3]{mp}-(6m+1)}.$$

Finally, Lemma 1.1 says that

$$\sqrt[3]{\vartheta_p(g)}+\sqrt[3]{\vartheta'_p(g)}+\sqrt[3]{\vartheta''_p(g)}=\sqrt[3]{S_p(g)}+\sqrt[3]{S'_p(g)}+\sqrt[3]{S''_p(g)},$$

so we are done. $\square$

1.2 Proof of Lemma 1.1

Again, let $p\equiv 1 \bmod 6$ be a prime number and let g be a primitive root mod p. The field element $g^3\in\mathbb{F}_p^\times$ generates the subgroup of $\mathbb{F}_p^\times$, which will be denoted by H. There are three cosets,

$$H=\{1,g^3,g^6,\dots,g^{p-7},g^{p-4}\}=\bigcup_{k=0}^{\frac{p-4}{3}}\{g^{3k}\},$$

$$gH=\{g,g^4,g^7,\dots,g^{p-6},g^{p-3}\}=\bigcup_{k=0}^{\frac{p-4}{3}}\{g^{3k+1}\},$$

$$g^2H=\{g^2,g^5,g^8,\dots,g^{p-5},g^{p-2}\}=\bigcup_{k=0}^{\frac{p-4}{3}}\{g^{3k+2}\}.$$

[As an aside note, the elements of H are called *cubic residues* mod p; the elements of $gH\sqcup g^2H$ are called *cubic non-residues* mod p.] To prove Lemma 1.1, we need two auxiliary propositions.

Proposition 1.1. *Let $S_p(g), S'_p(g), S''_p(g)$ be as above; then*

$$S_p(g) = \sum_{x \in H} \zeta^x, \quad S'_p(g) = \sum_{x \in gH} \zeta^x, \quad S''_p(g) = \sum_{x \in g^2 H} \zeta^x,$$

where $\zeta = e^{2\pi i/p}$, the primitive p-th root of unity.

PROOF. Consider two sets

$$I := \left\{1, g^3, g^6, \dots, g^{\frac{p-7}{2}}\right\} \quad \text{and} \quad J := \{g^{\frac{p-1}{2}}, g^{\frac{p+5}{2}}, \dots, g^{p-4}\};$$

then we have

$$H = \{\underbrace{1, g^3, g^6, \dots, g^{\frac{p-7}{2}}}_{\frac{p-1}{6}}, \underbrace{g^{\frac{p-1}{2}}, g^{\frac{p+5}{2}}, \dots, g^{p-4}}_{\frac{p-1}{6}}\} = I \sqcup J.$$

Now observe that any primitive root mod p is *a quadratic non-residue* mod p, hence

$$g^{\frac{p-1}{2}} \equiv -1 \bmod p.$$

Multiplying both sides by $g^3, g^6, \dots, g^{\frac{p-7}{2}}$, we get

$$g^{\frac{p+5}{2}} \equiv -g^3 \bmod p,$$

$$\dots\dots\dots$$

$$g^{p-4} \equiv -g^{\frac{p-7}{2}} \bmod p.$$

Note that there are all elements of J in the left-hand sides and the negatives of all elements of I in the right-hand sides. Because $\zeta^{n_1} = \zeta^{n_2}$ if and only if $n_1 \equiv n_2 \bmod p$, it follows that

$$\begin{cases} \zeta^{g^{\frac{p-1}{2}}} = \zeta^{-1}, \\ \zeta^{g^{\frac{p+5}{2}}} = \zeta^{-g^3}, \\ \dots\dots\dots \\ \zeta^{g^{p-4}} = \zeta^{-g^{\frac{p-7}{2}}}. \end{cases}$$

Addition of all this equalities gives

$$\sum_{x \in J} \zeta^x = \sum_{x \in I} \zeta^{-x};$$

next, we obtain

$$\sum_{x \in H} \zeta^x = \sum_{x \in I} \zeta^x + \sum_{x \in J} \zeta^x = \sum_{x \in I} \zeta^x + \sum_{x \in I} \zeta^{-x}.$$

Replacing x by $-x$ leads to

$$\sum_{x \in H} \zeta^{-x} = \sum_{x \in I} \zeta^{-x} + \sum_{x \in I} \zeta^x \quad \text{and thus} \quad \sum_{x \in H} \zeta^x = \sum_{x \in H} \zeta^{-x};$$

furthermore we have

$$\sum_{x \in H} \zeta^x = \frac{1}{2} \left(\sum_{x \in H} \zeta^x + \sum_{x \in H} \zeta^{-x} \right) = \sum_{x \in H} \cos\left(\frac{2\pi}{p} x\right).$$

Finally, since

$$S_p(g) := \sum_{k=0}^{\frac{p-4}{3}} \cos\left(\frac{2\pi}{p} g^{3k}\right) = \sum_{x \in H} \cos\left(\frac{2\pi}{p} x\right),$$

we see that

$$S_p(g) = \sum_{x \in H} \zeta^x.$$

An analysis like the one above shows that

$$S'_p(g) = \sum_{x \in gH} \zeta^x \quad \text{and} \quad S''_p(g) = \sum_{x \in g^2 H} \zeta^x.$$

□

Proposition 1.2. *Let g and $\tilde{g}$ be distinct primitive roots mod p; let H and $\tilde{H}$ be the subgroups of $\mathbb{F}_p^\times$ generated by g^3 and $\tilde{g}^3$, respectively. Then either*

$$\tilde{H} = H, \quad \tilde{g}\tilde{H} = gH, \quad \tilde{g}^2\tilde{H} = g^2H$$

or

$$\tilde{H} = H, \quad \tilde{g}\tilde{H} = g^2H, \quad \tilde{g}^2\tilde{H} = gH.$$

PROOF. First of all, we claim that $\tilde{g} \notin H$; let us assume the opposite and see what happens. Then $\tilde{g} = g^{3k}$ for some $k \in \mathbb{Z}$ and hence $\tilde{g}^l$ belongs to H for any $l \in \mathbb{Z}$; therefore the field element $\tilde{g}$ cannot generate $\mathbb{F}_p^\times$, a contradiction. Accordingly, either $\tilde{g} \in gH$ or else $\tilde{g} \in g^2H$.

- Assume that $\tilde{g} \in gH$; then $\tilde{g} = g^{3k+1}$ for some $k \in \mathbb{Z}$ and thus we have

$$\tilde{H} = \bigcup_{l=0}^{\frac{p-4}{3}} \{\tilde{g}^{3l}\} = \bigcup_{l=0}^{\frac{p-4}{3}} \{g^{(3k+1)3l}\},$$

$$\tilde{g}\tilde{H} = \bigcup_{l=0}^{\frac{p-4}{3}} \{\tilde{g}^{3l+1}\} = \bigcup_{l=0}^{\frac{p-4}{3}} \{g^{(3k+1)(3l+1)}\} = \bigcup_{l=0}^{\frac{p-4}{3}} \{g^{9kl+3(k+l)+1}\},$$

$$\tilde{g}^2\tilde{H} = \bigcup_{l=0}^{\frac{p-4}{3}} \{\tilde{g}^{3l+2}\} = \bigcup_{l=0}^{\frac{p-4}{3}} \{g^{(3k+1)(3l+2)}\} = \bigcup_{l=0}^{\frac{p-4}{3}} \{g^{9kl+3(k+l)+2}\}.$$

  Since $\tilde{H}$ consists of elements of the form g^{3n} for some $n \in \mathbb{Z}$ and $|\tilde{H}| = |H|$, we conclude that $\tilde{H} = H$; the same argument shows that $\tilde{g}\tilde{H} = gH$ and $\tilde{g}^2\tilde{H} = g^2H$.

- Assume that $\tilde{g} \in g^2H$; in the same way one can check that

$$\tilde{H} = H, \quad \tilde{g}\tilde{H} = g^2H, \quad \tilde{g}^2\tilde{H} = gH.$$

□

As we have already mentioned, the group $\mathbb{F}_p^\times$ has exactly two characters of order 3. Now we can say more: these characters are called *cubic characters* mod p and denoted by χ_{cub} and $\bar{\chi}_{\text{cub}}$. They are complex conjugate and given explicitly by

$$\chi_{\text{cub}}(x) = \begin{cases} 1, \text{ if } x \in H \\ \omega, \text{ if } x \in gH \\ \omega^2, \text{ if } x \in g^2H, \end{cases} \qquad \bar{\chi}_{\text{cub}}(x) = \begin{cases} 1, \text{ if } x \in H \\ \omega^2, \text{ if } x \in gH \\ \omega, \text{ if } x \in g^2H, \end{cases}$$

where $\omega = e^{2\pi i/3}$, the primitive cube root of unity.

PROOF OF LEMMA 1.1 . Let us prove that either

$$\vartheta_p(\chi) = S_p(g), \quad \vartheta_p'(\chi) = S_p'(g), \quad \vartheta_p''(\chi) = S_p''(g)$$

or

$$\vartheta_p(\chi) = S_p(g), \quad \vartheta_p'(\chi) = S_p''(g), \quad \vartheta_p''(\chi) = S_p'(g),$$

i.e., a different choice of χ would permute $\vartheta_p'(\chi)$ and $\vartheta_p''(\chi)$ among themselves.

- If $\chi = \chi_{\text{cub}}$ then we have

$$\vartheta_p(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=1}} \zeta^x = \sum_{x \in H} \zeta^x, \quad \vartheta_p'(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=\omega}} \zeta^x = \sum_{x \in gH} \zeta^x, \quad \vartheta_p''(\chi) := \sum_{\substack{x \in \mathbb{F}_p^\times \\ \chi(x)=\omega^2}} \zeta^x = \sum_{x \in g^2H} \zeta^x.$$

  Now Proposition 1.1 tell us that

$$\vartheta_p(\chi) = S_p(g), \quad \vartheta_p'(\chi) = S_p'(g), \quad \vartheta_p''(\chi) = S_p''(g).$$

- If $\chi = \bar{\chi}_{\text{cub}}$ then arguing as above one can verify that

$$\vartheta_p(\chi) = S_p(g), \quad \vartheta_p'(\chi) = S_p''(g), \quad \vartheta_p''(\chi) = S_p'(g).$$

  as needed.

On the other hand, the values of $S_p(g)$, $S'_p(g)$, $S''_p(g)$ depend on the choice of g. Although there are $\phi(p-1)$ primitive roots mod p, where ϕ denotes Euler's totient function, we must only check that if we replace g by other primitive root, say $\tilde{g}$, then S'_p and S''_p will switch places. To see this, we have to consider two cases.

- Assume that $\tilde{g} \in gH$; using Proposition 1.2, we find

$$\sum_{x\in \tilde{H}} \zeta^x = \sum_{x\in H} \zeta^x, \quad \sum_{x\in \tilde{g}\tilde{H}} \zeta^x = \sum_{x\in gH} \zeta^x, \quad \sum_{x\in \tilde{g}^2\tilde{H}} \zeta^x = \sum_{x\in g^2H} \zeta^x.$$

  From Proposition 1.1 it follows that

$$S_p(\tilde{g}) = \sum_{x\in \tilde{H}} \zeta^x, \quad S'_p(\tilde{g}) = \sum_{x\in \tilde{g}\tilde{H}} \zeta^x, \quad S''_p(\tilde{g}) = \sum_{x\in \tilde{g}^2\tilde{H}} \zeta^x;$$

  in this way we obtain

$$S_p(\tilde{g}) = S_p(g), \quad S'_p(\tilde{g}) = S'_p(g), \quad S''_p(\tilde{g}) = S''_p(g).$$

- Assume that $\tilde{g} \in g^2H$; the same argument as above yields

$$S_p(\tilde{g}) = S_p(g), \quad S'_p(\tilde{g}) = S''_p(g), \quad S''_p(\tilde{g}) = S'_p(g).$$

This completes the proof of the lemma. □

1.3 Remarks

Let $\mathsf{Primes}_1 := \{\text{all primes } p \equiv 1 \bmod 6\}$ and let $\mathsf{Primes}_2 := \{\text{all primes } p = 9m^2+3m+1 \text{ for some } m \in \mathbb{Z}\ \}$; it is clear that $\mathsf{Primes}_2 \subset \mathsf{Primes}_1$. According to Dirichlet's theorem on arithmetic progressions, the set Primes_1 is infinite; but for the best of my knowledge, it is an open question whether the set Primes_2 is infinite or not.

Table 1: all primes $p \in \mathsf{Primes}_2$ up to 1000.

p	$h(\vartheta)$	m	g
7	$\vartheta^3+\vartheta^2-2\vartheta-1$	-1	3
13	$\vartheta^3+\vartheta^2-4\vartheta+1$	1	2
31	$\vartheta^3+\vartheta^2-10\vartheta-8$	-2	3
43	$\vartheta^3+x^2-14x+8$	2	3
73	$\vartheta^3+\vartheta^2-24\vartheta-27$	-3	5
157	$\vartheta^3+\vartheta^2-52\vartheta+64$	4	5
211	$\vartheta^3+\vartheta^2-70\vartheta-125$	-5	2
241	$\vartheta^3+\vartheta^2-80\vartheta+125$	5	7
307	$\vartheta^3+\vartheta^2-102\vartheta-216$	-6	5
421	$\vartheta^3+\vartheta^2-140\vartheta-343$	-7	2
463	$\vartheta^3+\vartheta^2-154\vartheta+343$	7	3
601	$\vartheta^3+\vartheta^2-200\vartheta+512$	8	7
757	$\vartheta^3+\vartheta^2-252\vartheta+729$	9	2

Using Table 1 one can construct further examples of Ramanujan-type trigonometric identities. For instance, taking $p=757$ we get

$$\sqrt[3]{\sum_{k=0}^{251}\cos\left(\frac{2\pi}{757}\cdot 2^{3k}\right)} + \sqrt[3]{\sum_{k=0}^{251}\cos\left(\frac{2\pi}{757}\cdot 2^{3k+1}\right)} + \sqrt[3]{\sum_{k=0}^{251}\cos\left(\frac{2\pi}{757}\cdot 2^{3k+2}\right)} = \sqrt[3]{3\sqrt[3]{6813}-55}.$$

2 Elliptic curves

2.1 Proof of Theorem 2 modulo Lemmas 2.1, 2.2

The following result was supplied by M.Zieve via the MathOverflow site, see [mo243730].

Theorem. (Zieve) *Let x_1, x_2, x_3 be the complex roots of the cubic x^3+rx+q with rational coefficients. Suppose that the curve $E_0\colon Y^2 = X^3 + \frac{27}{4}\cdot(4r^3+27q^2)$ has infinitely many rational points. Then there are infinitely many pairs of rational numbers (u,v) satisfying the equation*

$$(u-x_1)^{\frac{1}{3}} + (u-x_2)^{\frac{1}{3}} + (u-x_3)^{\frac{1}{3}} = \sqrt[3]{v} \tag{3}$$

(*for some choice of complex cube roots in the left-hand side*).

This brings us to the following questions.

- **Question 1:** *For which values of* (r, q) *does the curve* $E_0\colon Y^2 = X^3 + \frac{27}{4} \cdot (4r^3 + 27q^2)$ *have infinitely many rational points?*
- Suppose given equation (3) with $x_1, x_2, x_3 \in \mathbb{R}$ are all real; or, what is the same, given the equation

$$\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3} \cdot \omega^{l_3} = \sqrt[3]{v},$$

  where ω is a primitive cube root of unity and $l_1, l_2, l_3 \in \{0, 1, 2\}$.

  Question 2: *Can we claim that* $(l_1, l_2, l_3) = (0, 0, 0)$ *?*

The following statement provides an answer to the first question.

Lemma 2.1. *Let* r, q *be any rational numbers such that*

$$r \neq 0, \quad q \neq 0, \quad r^3 + 9q^2 \neq 0, \quad r^3 + 6q^2 \neq 0.$$

Then the curve $E_0\colon Y^2 = X^3 + \frac{27}{4} \cdot (4r^3 + 27q^2)$ *has infinitely many rational points.*

To handle Question 2 we proceed as follows.

(a) It suffices to show that the complex number

$$\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3} \cdot \omega^{l_3}$$

has nonzero imaginary part unless $(l_1, l_2, l_3) = (0, 0, 0)$.

(b) If the cube of an arbitrary complex number has nonzero imaginary part, then the complex number also has nonzero imaginary part. It thus suffices to show that the complex number

$$\left(\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3} \cdot \omega^{l_3}\right)^3$$

has nonzero imaginary part unless $(l_1, l_2, l_3) = (0, 0, 0)$, $(1, 1, 1)$, $(2, 2, 2)$.

(c) Since multiplying each summand in the brackets by ω doesn't change the value of

$$\left(\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3} \cdot \omega^{l_3}\right)^3,$$

it suffices to show that the complex number

$$\left(\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3}\right)^3$$

has nonzero imaginary part unless $(l_1, l_2) = (0, 0)$.

Thus, Lemma 2.2 gives an affirmative answer to the second question.

Lemma 2.2. *Let* x_1, x_2, x_3 *be the real roots of the cubic* $x^3 + rx + q$ *with rational coefficients,* $u \in \mathbb{Q}$ *be any rational number,* ω *be a primitive cube root of unity, and* $l_1, l_2, l_3 \in \{0, 1, 2\}$*. Then the complex number*

$$\left(\sqrt[3]{u - x_1} \cdot \omega^{l_1} + \sqrt[3]{u - x_2} \cdot \omega^{l_2} + \sqrt[3]{u - x_3}\right)^3$$

has nonzero imaginary part unless $(l_1, l_2) = (0, 0)$.

The proof of Theorem 2 will rest on the following facts.

(i) Given a prime number $p \equiv 1 \bmod 6$ and the representation $4p = a^2 + 27b^2$, where $a \equiv 1 \bmod 3$ and $b > 0$; then the real numbers

$$\eta_p(\chi) := 3\vartheta_p(\chi) + 1, \quad \eta'_p(\chi) := 3\vartheta'_p(\chi) + 1, \quad \eta''_p(\chi) := 3\vartheta''_p(\chi) + 1$$

are roots of the polynomial $\eta^3 - 3p\eta - ap \in \mathbb{Z}[\eta]$, see [Has, p.461].

(ii) Let $\tilde{\alpha}, \tilde{\beta}$ be rational numbers such that

$$\sqrt[3]{\tilde{\alpha} - \eta_p(\chi)} + \sqrt[3]{\tilde{\alpha} - \eta'_p(\chi)} + \sqrt[3]{\tilde{\alpha} - \eta''_p(\chi)} = \sqrt[3]{\tilde{\beta}};$$

then one can check by direct calculation that the rational numbers $\alpha := \dfrac{\tilde{\alpha} - 1}{3}, \beta := \dfrac{\tilde{\beta}}{3}$ satisfy the equation

$$\sqrt[3]{\alpha - \vartheta_p(\chi)} + \sqrt[3]{\alpha - \vartheta'_p(\chi)} + \sqrt[3]{\alpha - \vartheta''_p(\chi)} = \sqrt[3]{\beta}.$$

PROOF OF THEOREM 2. Given the polynomial $\eta^3 - 3p\eta - ap$; in the notation of Zieve's theorem, we have

$$r = -3p \quad \text{and} \quad q = -ap.$$

In order to apply Lemma 2.1, we must show that

$$r^3 + 9q^2 \neq 0 \quad \text{and} \quad r^3 + 6q^2 \neq 0$$

or equivalently, that

$$3p \neq a^2 \quad \text{and} \quad 9p \neq 2a^2.$$

Since $a \equiv 1 \bmod 3$, this is indeed the case. Now the proof is in five steps:

1. Lemma 2.1 implies that there are infinitely many rational points on the curve $E_0\colon Y^2 = X^3 + \frac{27}{4} \cdot (4r^3 + 27q^2)$, where $r = -3p$ and $q = -ap$.
2. Zieve's theorem implies that there are infinitely many pairs of rational numbers (u, v) satisfying the equation

   $$(u - \eta_p(\chi))^{\frac{1}{3}} + (u - \eta_p'(\chi))^{\frac{1}{3}} + (u - \eta_p''(\chi))^{\frac{1}{3}} = \sqrt[3]{v}$$

   (for some choice of complex cube roots in the left-hand side).
3. Lemma 2.2 implies that there are infinitely many pairs of rational numbers $(\tilde{\alpha}, \tilde{\beta})$ such that

   $$\sqrt[3]{\tilde{\alpha} - \eta_p(\chi)} + \sqrt[3]{\tilde{\alpha} - \eta_p'(\chi)} + \sqrt[3]{\tilde{\alpha} - \eta_p''(\chi)} = \sqrt[3]{\tilde{\beta}}.$$

4. Fact (ii) implies that there are infinitely many pairs of rational numbers (α, β) such that

   $$\sqrt[3]{\alpha - \vartheta_p(\chi)} + \sqrt[3]{\alpha - \vartheta_p'(\chi)} + \sqrt[3]{\alpha - \vartheta_p''(\chi)} = \sqrt[3]{\beta}, \quad \text{where } \alpha = \frac{\tilde{\alpha} - 1}{3} \text{ and } \beta = \frac{\tilde{\beta}}{3}.$$

5. Lemma 1.1 implies that there are infinitely many pairs of rational numbers (α, β) such that

   $$\sqrt[3]{\alpha - S_p(g)} + \sqrt[3]{\alpha - S_p'(g)} + \sqrt[3]{\alpha - S_p''(g)} = \sqrt[3]{\beta}.$$

□

2.2 Proof of Lemmas 2.1, 2.2

PROOF OF LEMMA 2.1. Suppose that $4r^3 + 27q^2 = 0$; then one can get infinitely many rational points on the curve $E_0\colon Y^2 = X^3$ via a rational parametrization. In what follows we assume that $4r^3 + 27q^2 \neq 0$; then E_0 is a nonsingular plane cubic curve with the point $M := (-3r : \frac{27}{2}q : 1)$—that is, an **elliptic curve**. The order of M in the group $E_0(\mathbb{Q})$ can be 2, 3, or 6; see, e.g., [Kn, p.134, Th.5.3]. Using standard computer software one can obtain

$$2M = \left(rq(r^3 + 6q^2) : -(r^6 + 9r^3q^2 + \frac{27}{2}q^4) : q^3 \right),$$

$$3M = \left(-3r(r^3 + 9q^2)(r^9 - 81r^3q^4 - 243q^6) : \frac{81}{2}q(r^3 + 6q^2)(r^9 + 9r^6q^2 + 27r^3q^4 + 81q^6) : r^3(r^3 + 9q^2)^3 \right),$$

$$\begin{aligned} 6M = \Big(& \frac{1}{9}qr(r^3 + 6q^2)(r^3 + 9q^2)(r^9 + 9r^6q^2 + 27r^3q^4 + 81q^6)(r^9 - 81r^3q^4 - 243q^6)(r^{27} + 54r^{24}q^2 + 1377r^{21}q^4 + \\ & 20169r^{18}q^6 + 177147r^{15}q^8 + 918540r^{12}q^{10} + 2558790r^9q^{12} + 2657205r^6q^{14} - 1594323r^3q^{16} - 1594323q^{18}) : \\ & -\frac{1}{27}(r^6 + 9r^3q^2 + \frac{27}{2}q^4)(r^{12} + 18r^9q^2 + 108r^6q^4 - 1458q^8)(r^{36} + 54r^{33}q^2 + 1296r^{30}q^4 + 24300r^{27}q^6 + \\ & 426465r^{24}q^8 + 5799924r^{21}q^{10} + 52986636r^{18}q^{12} + 310361544r^{15}q^{14} + 1123997715r^{12}q^{16} + 2324522934r^9q^{18} + \\ & 2238429492r^6q^{20} + 516560652r^3q^{22} + 387420489q^{24}) : q^3r^3(r^3 + 6q^2)^3(r^3 + 9q^2)^3(r^9 + 9r^6q^2 + 27r^3q^4 + 81q^6)^3 \Big). \end{aligned}$$

Let us now prove that the point M is of **infinite order** in the group $E_0(\mathbb{Q})$.

- If $2M = O$, where $O = (0 : 1 : 0)$ is the point at infinity, then

  $$\left(rq(r^3 + 6q^2) : -(r^6 + 9r^3q^2 + \frac{27}{2}q^4) : q^3 \right) \sim (0 : 1 : 0),$$

  which contradicts to the assumption that $q \neq 0$.

- Similarly, the equality $3M = O$ implies that
$$r^3(r^3+9q^2)^3 = 0;$$
this contradicts to the assumptions that $r \neq 0$ and $r^3 + 9q^2 \neq 0$.
- The equality $6M = O$ implies that
$$q^3r^3(r^3+6q^2)^3(r^3+9q^2)^3(r^9+9r^6q^2+27r^3q^4+81q^6)^3 = 0.$$
To complete the proof of the lemma, it is enough to show that the equation
$$r^9+9r^6q^2+27r^3q^4+81q^6 = 0$$
has no solutions over $\mathbb{Q}$. On the contrary, suppose that there exists such a rational solution (r,q); then $(\tilde{r},\tilde{q}) := (r^3+3q^2, -3q^2)$ is a rational solution of the equation $\tilde{r}^3 = 2\tilde{q}^3$, which is a contradiction.

□

PROOF OF LEMMA 2.2. For the sake of brevity, take
$$y_1 := u - x_1, \quad y_2 := u - x_2, \quad y_3 := u - x_3.$$
Since $\mathbb{R}[\omega]$ is a degree 2 extension of $\mathbb{R}$, one can write
$$\sqrt[3]{y_1}\cdot\omega^{l_1} + \sqrt[3]{y_2}\cdot\omega^{l_2} + \sqrt[3]{y_3} = z_1 + z_2\cdot\omega \quad \text{for some } z_1, z_2 \in \mathbb{R};$$
thence
$$(\sqrt[3]{y_1}\cdot\omega^{l_1} + \sqrt[3]{y_2}\cdot\omega^{l_2} + \sqrt[3]{y_3})^3 = z_1^3 + z_2^3 - 3z_1z_2^2 + 3z_1z_2(z_1 - z_2)\cdot\omega.$$
To establish the claim it thus suffices to show that $z_1z_2(z_1-z_2) \neq 0$. To illustrate the idea, consider only the case when $(l_1, l_2) = (2,2)$; the analysis of the other cases is similar and left to the reader. One has
$$\sqrt[3]{y_1}\cdot\omega^2 + \sqrt[3]{y_2}\cdot\omega^2 + \sqrt[3]{y_3} = z_1 + z_2\omega,$$
with
$$z_1 = \sqrt[3]{y_3} - \sqrt[3]{y_2} - \sqrt[3]{y_1} \quad \text{and} \quad z_2 = -\sqrt[3]{y_1} - \sqrt[3]{y_2}.$$
Let us prove by contradiction—suppose that $z_1z_2(z_1-z_2) = 0$.

- If $z_1 = 0$ then $\sqrt[3]{y_1} + \sqrt[3]{y_2} = \sqrt[3]{y_3}$; cubing both sides and simplifying, we get
$$3\sqrt[3]{y_1y_2y_3} = 2y_3 - (y_1+y_2+y_3)$$
whence
$$y_3 = \frac{3\sqrt[3]{y_1y_2y_3} + y_1 + y_2 + y_3}{2}.$$
Since x_1, x_2, x_3 are roots of the polynomial $f(x) := x^3 + rx + q$, it follows that y_1, y_2, y_3 are roots of the polynomial
$$-f(u-x) = x^3 - 3ux^2 + \ldots$$
Using Vieta's formulas, we find
$$y_1 + y_2 + y_3 = 3u \quad \text{and} \quad y_1y_2y_3 = f(u);$$
hence,
$$y_3 = \frac{3}{2}\cdot\left(\sqrt[3]{f(u)} + u\right).$$
The minimal polynomial of the number in the right-hand side is
$$\frac{27}{8}\cdot\left(\left(\frac{2x}{3} - u\right)^3 - f(u)\right) = x^3 - \frac{9ux^2}{2} + \ldots,$$
which is a contradiction because $x^3 - 3ux^2 + \ldots \not\equiv x^3 - \frac{9ux^2}{2} + \ldots$ in $\mathbb{Q}[x]$.
- If $z_2 = 0$ then $y_1 = -y_2$, which contradicts to the fact that $y_1 + y_2 + y_3 = 3u$.
- If $z_1 - z_2 = 0$ then $y_3 = 0$—contradiction.

□

2.3 Remarks

M.Zieve also established the following result.

There exist rational functions $u(X,Y), v(X,Y) \in \mathbb{Q}(X,Y)$ such that the following holds:
Let (X_0, Y_0) be any rational point on the curve $E_0\colon Y^2 = X^3 + \frac{27}{4}\cdot(4r^3+27q^2)$ such that the functions $u(X,Y), v(X,Y)$ are defined at the point (X_0,Y_0). Then the pair of rational numbers $(u(X_0,Y_0), v(X_0,Y_0))$ satisfy equation (3).

Some solutions are compiled in the table below whereas others are not compiled due to lack of space.

Table 2: points on E_0 and solutions of equation (3)

points on $E_0\colon Y^2 = X^3 + \frac{27}{4}\cdot(4r^3+27q^2)$	u	v
$-2M = (rq(r^3+6q^2) : (r^6+9r^3q^2+\frac{27}{2}q^4) : q^3)$	$\frac{r^{12}-135r^6q^4-729r^3q^6-729q^8}{9rq(r^3+6q^2)(r^6+9r^3q^2+27q^4)}$	$\frac{3r^2(r^3+6q^2)^2}{r^6q+9r^3q^3+27q^5}$
$-M = (-3r : -\frac{27}{2}q : 1)$	$-\frac{q}{r}$	0
$M = (-3r : \frac{27}{2}q : 1)$	$\frac{-r^6q+27q^5}{r^7+9r^4q^2+27rq^4}$	$\frac{27r^2q^3}{r^6+9r^3q^2+27q^4}$

Using Table 2, we get

$$\sqrt[3]{-\frac{q}{r}-\eta_p(g)}+\sqrt[3]{-\frac{q}{r}-\eta_p'(g)}+\sqrt[3]{-\frac{q}{r}-\eta_p''(g)}=0, \qquad \text{with } r=-3p \text{ and } q=-ap.$$

Now fact (ii) and Lemma 1.1 together imply that

$$\sqrt[3]{\alpha_0-S_p(g)}+\sqrt[3]{\alpha_0-S_p'(g)}+\sqrt[3]{\alpha_0-S_p''(g)}=0, \qquad \text{with } \alpha_0=-\frac{a+3}{9}.$$

Since the function $\mathbb{R}\to\mathbb{R}$ given by $\alpha\mapsto\sqrt[3]{\alpha-S_p(g)}+\sqrt[3]{\alpha-S_p'(g)}+\sqrt[3]{\alpha-S_p''(g)}$ is strictly increasing and continuous, there exists a unique $\hat{\alpha}\in\mathbb{R}$ at which this function vanishes. It happens that such an $\hat{\alpha}$ is rational.

References

[AR] Ireland K., Rosen M. I. A Classical Introduction to Modern Number Theory. – Springer Science & Business Media, 1990.

[BCK] Berndt B. C., Choi Y. S., Kang S. Y. The Problems Submitted by Ramanujan to the Journal of the Indian Mathematical Society.

[Has] Hasse H. Vorlesungen über Zahlentheorie, 1950.

[Lev] Левин В. И. Рамануджан – математический гений Индии. М., Знание, 1968. [in Russian]

[Mar] S. V. Markelov, “О тождествах Рамануджана”, Mat. Pros., Ser. 3, 9, MCCME, Moscow, 2005, 218–219 [in Russian], available at http://mi.mathnet.ru/eng/mp/v9/s3/p218

[dxdy54521] S. V. Markelov, Тождество с cos(Pi/19) и кубическими корнями. [in Russian], URL (version: 2012-01-30): http://dxdy.ru/topic54521.html

[ms1842663] mercio, What's the explanation for these (infinitely many?) Ramanujan-type identities?, URL (version: 2016-06-29): https://math.stackexchange.com/q/1842663

[mo243730] Michael Zieve, An elliptic curve for Ramanujan-type cubic identities?, URL (version: 2016-07-06): https://mathoverflow.net/q/243730

[Pr] V. V. Prasolov, “Тождества Рамануджана”, Mat. Pros., Ser. 3, 9, MCCME, Moscow, 2005, 104–107 [in Russian], available at http://mi.mathnet.ru/eng/mp/v9/s3/p104

[Sh] V. S. Shevelev, “Три формулы Рамануджана.”, Kvant, No. 6, 52–55 (1988) [in Russian], available at http://kvant.mccme.ru/1988/06/tri_formuly_ramanudzhana.htm

[Kn] Knapp, Anthony W. Elliptic curves. Vol. 40. Princeton University Press, 1992.